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RIGIDITY THEOREMS FOR RIGHT ANGLED REFLECTION GROUPS

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ABSTRACT. Let Γ be a right angled reflection group. Let M and M' be Coxeter manifolds. Then any Γ-map $f: M \to M'$ is Γ-homotopic to a homeomorphism

Introduction. This paper contains the results given in the author's Ph.D. thesis.

Davis in [**9**], showed that in every dimension ≥ 4, there exists a cocompact reflection group on a contractible manifold not homeomorphic to an Euclidean space. Also he gave the first example of a closed aspherical manifold not covered by Euclidean space.

Davis construction provides an infinite Coxeter group acting locally smoothly, effectively and properly discontinuously on a contractible manifold with compact quotient.

The construction is as follows: Let K be a finite simplicial complex that is a generalized homology sphere. Let V be the set of vertices of K' (the baricentric subdivision of K).

So $V = \{\sigma' | \sigma \text{ is a simplex of } K\}$. Define $m: V \times V \to \{1, 2, 3, 4, \ldots, \infty\}$ by $m(\sigma', \tau') = 1$ if $\sigma = \tau$, 2 if $\sigma < \tau$ or $\tau < \sigma$, and ∞ otherwise. This defines a Coxeter group (Γ, V) (see 1.1).

Let K'' be the dual polyhedron of K. There is a recognized fact in surgery theory that there exists a contractible compact manifold X^n, $n \geq 4$, such that $\partial X = K''$. Give to X the following V-panelled structure $(X_{\sigma'})_{\sigma' \in V}$, where $X_{\sigma'}$ is defined as the dual cell of σ'. So X_S is $\varnothing$ if $S = \{\sigma'_1, \sigma'_2, \sigma'_k\}$ is not a simplex of K' or the dual $n-k$ cell of S if S is a k-simplex. In this form $\{X, (X_{\sigma'})_{\sigma' \in V}\}$ is a contractible V-panelled manifold (see [**9**]), and each face of X is contractible.

Form the universal space $U(\Gamma, X)$ (see 1.5). By [**9**, Theorem 10.1], $U(\Gamma, X)$ is contractible, and by 1.12, $U(\Gamma, X)$ is a manifold. Also Γ acts as cocompact reflection group.

All the Davis examples on which $U(\Gamma, X) \neq R^n$, $n \geq 4$, come from "right angled Coxeter groups" (this means that the only entries of the Coxeter matrix are 1, 2, and ∞).

The construction has the additional attractive property that $U(\Gamma, X)^H$ is contractible if H is any finite subgroup of Γ (see 2.12).

These Γ-manifolds $U(\Gamma, X)$ motivate the definition of Coxeter manifolds. A Coxeter manifold is a Coxeter group Γ and a manifold N on which Γ acts in a locally smooth, effective and properly discontinuous fashion such that N^H is contractible

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for each finite subgroup H of Γ, N/Γ is compact and if H is a finite subgroup of Γ such that N^H has dimension three, then $N^H \approx R^3$.

The purpose of this thesis is to prove the following conjecture: If (M,Γ) and (M',Γ) are Coxeter manifolds, then M and M' are equivariantly homeomorphic.

A Coxeter group (Γ, V) is said to be a right angled reflection group if Γ acts on a contractible manifold Y^n with compact quotient and satisfies the following conditions:

(1) The generators $v \in V$ acts as reflections on Y.

(2) The action is locally smooth, effective and properly discontinuous.

(3) The only entries of the Coxeter matrix are 1, 2 and ∞.

(4) If $V \supset S$, such that S generates a finite subgroup G of Γ, then Y^G is contractible.

The main theorem.

THEOREM 3.11 (RIGIDITY THEOREM FOR COXETER MANIFOLDS). *Let (M,Γ) and (M',Γ) be two Coxeter manifolds. Assume that Γ is a right angled reflection group. Then M is equivariantly homeomorphic to M'. In fact, any Γ-map $f\colon M \to M'$ is Γ-homotopic to a homeomorphism.*

There is evidence that the right angled reflection group hypothesis is unnecessary.

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CHAPTER 1. BASIC DEFINITIONS AND THEOREMS

Coxeter groups. Let Γ be a group and V be a set of generators, each element of which has order two. For any pair of elements (s,s') of V, denote by $m(s,s')$ the order of ss', and let I be the set of pairs (s,s') of distinct elements in V such that $m(s,s')$ is finite.

DEFINITION 1.1 [**1**]. The pair (Γ, V) is a Coxeter group if the set of generators V together with the relations

$s^2 = 1, \; s \in V,$

$(ss')^{m(s,s')} = 1$, where $(s,s') \in I$

form a presentation for Γ.

If the order of ss' is two or infinite, for all $(s,s') \in I$, we say that (Γ, V) is a right angled Coxeter group.

Let (Γ, V) be a Coxeter group and $V \supseteq S$. Denote by Γ_s the subgroup of Γ generated by S.

PROPOSITION 1.2 [**1**, p. 19]. *Let $g \in \Gamma$. There exists a subset V_g of V such that for each reduced decomposition $(s_1, s_2, \ldots, s_n)$ of g, (see [**1**] for the definition) $V_g = \{s_1, s_2, \ldots, s_n\}$. If S is a subset of V and $g \in \Gamma_s$, then $S \supset V_g$.*

Panel structure.

DEFINITION 1.3 [**9**]. A panel structure on a topological space Q is a locally finite family of closed subspaces $(Q_v)_{v\in V}$, indexed by some set V. The Q_v are the panels of Q. A space together with a panel structure is called a V-panelled space, and we denote it by $\{Q, (Q_v)_{v\in V}\}$.

For each $x \in Q$, let $V(x)$ denote the set of v in V such that $x \in Q_v$. For each subset S of V denote by Q_S the set of x in Q such that $V(x)$ contains S. The Q_S are the faces of Q. The formal boundary of Q is defined as: $\partial Q = \bigcup\{Q_v \colon v \in V\}$.

Note that $Q_\varnothing = Q$, and $Q_S = \bigcap_{v\in S} Q_v$.

If S is a subset of V, we further define $Q_{\sigma(S)} = \bigcup_{v\in S} Q_v$.

Observe that if A is a subspace of a V-panelled space $\{Q, (Q_v)_{v\in V}\}$, then $\{A, (A_v)_{v\in V}\}$ is also a V-panelled space, where the panel structure $(A_v)_{v\in V}$ is defined by the equation $A_v = A \cap Q_v$.

DEFINITION 1.4. A map of panelled spaces $\{Q, (Q_v)_{v\in V}\}$ and $\{Q', (Q'_w)_{w\in W}\}$ is a continuous function $\Phi: Q \to Q'$ such that for each $v \in V$ with $Q_v \neq \varnothing$, there exists $w \in W$ such that $Q'_w \supseteq \Phi(Q_v)$.

A homeomorphism of panelled spaces is an isomorphism in this category.

The universal space.

DEFINITION 1.5 [**17**]. Let (Γ, V) be a Coxeter group and $\{Q, (Q_v)_{v\in V}\}$ be a V-panelled space. The panel structure of Q is called Γ-finite if $\Gamma_{V(x)}$ is finite for all $x \in Q$. (See 1.3 for the definition of $V(x)$.)

Let (Γ, V) be a Coxeter group and $\{Q, (Q_v)_{v\in V}\}$ be a V-panelled space. Give Γ the discrete topology. Consider the following equivalence relation on the topological product $\gamma \times Q$:

$$(\alpha, x) \sim (\beta, y) \quad \text{if and only if } x = y \text{ and } \beta\alpha^{-1} \in \Gamma_{V(x)}.$$

The universal space $U(\Gamma, Q)$ is defined as the factor space $\Gamma \times Q/\sim$.

Let $\pi: \Gamma \times Q \to U(\Gamma, Q)$ be a canonical projection defined by

$$\pi((\alpha, x)) = [\alpha, x].$$

The action of Γ on $\Gamma \times Q$ is defined by $\alpha(\beta, y) = (\alpha\beta, y)$. This action is compatible with the equivalence relation $\sim$. Hence it induces an action of Γ on $U(\Gamma, Q)$, namely $\alpha[\beta, x] = [\alpha\beta, x]$.

Note that Q embeds in $U(\Gamma, Q)$ via $x \to [1, x]$.

Since for each $[1, x] \in U(\Gamma, Q)$, the isotropy group is $\Gamma_{V(x)}$, for each $[g, x] \in U(\Gamma, Q)$, its isotropy group is $g\Gamma_{V(x)}g^{-1}$.

$U(\Gamma, Q)$ is decomposed as a union of the sets of the form αQ_S, $V \supseteq S$. These will be called faces. The intersection of any two faces of $U(\Gamma, Q)$ is a third face (or is empty); one sees this immediately from the following lemma.

LEMMA 1.6. *Let $g \in \Gamma$ and V_g be the subset of V defined in Proposition* 1.2. *Then $gQ_S \cap Q_T = Q_{S\cup T\cup V_g}$.*

PROOF. This follows immediately from the definition of $\sim$.

PROPOSITON 1.7 [**17**, p. 1088]. *Let Y be a Γ space and $f: Q \to Y$ be a map such that $vf(x) = f(x)$ for all $v \in V$ and $x \in Q_v$. Then there is a unique Γ-equivariant map $f^*: U(\Gamma, Q) \to Y$ such that $f^*([1, x]) = f(x)$ for all $x \in Q$.*

In particular, letting $Y = U(\Gamma, Q')$ we see from 1.5 *that a map $f: Q \to Q'$ of V-panelled spaces defines a Γ-equivariant map $U(\Gamma, f): (\Gamma, Q) \to U(\Gamma, Q')$ with the above functorial properties.*

PROPOSITION 1.8 [**17**, p. 1088]. *Suppose Q is a Hausdorff space. Then Γ acts properly on $U(\Gamma, Q)$ if and only if the panel structure on Q is Γ-finite.*

Reflection groups. Let M be a connected manifold. A reflection $r: M \to M$ is a locally smooth involution such that the fixed set M^r separates M into exactly two path components.

DEFINITION 1.9. Suppose that Γ is a discrete group acting properly, locally smoothly and effectively on a connected manifold M, and that Γ is generated by reflections on M. Then Γ is called a reflection group on M.

Let Γ be a reflection group on M, and let R denote the set of all reflections in Γ. For each $x \in M$, let $R(x)$ be the set of all $r \in R$ such that x belongs to M^r. A point x is called nonsingular if $R(x) = \varnothing$. A chamber of Γ on M is defined as the closure of a connected component of the set of nonsingular points.

Let Q be a chamber. Denote by V the set of reflections v such that for some $x \in Q$, $R(x) = \{v\}$. We now define a panel structure on Q as follows: for each $v \in V$, set $Q_v = M^v \cap Q$.

THEOREM 1.10 [**9**, p. 301]. *Let Γ be a reflection group on a manifold M. Let Q be a chamber and V be the set of reflections defined above. The following statements are true*:

1. *(Γ, V) is a Coxeter group. In particular, V generates Γ.*
2. *For each $v \in V$ and each $g \in \Gamma$ the relation $L(vg) \geq L(g)$ means that Q and gQ are on the same side of the wall M^v. (See [1] for the definition of $L(g)$.)*
3. *Γ acts freely and transitively on the set of chambers in M.*
4. *If $x, y \in Q$ and $gx = y$ for some $g \in \Gamma$, then $x = y$ and $g \in \Gamma_{V(x)}$.*
5. *Q is a closed fundamental domain; i.e. if $\pi: M \to M/\Gamma$ denotes the orbit map then $\pi|_Q$ is a homeomorphism.*
6. *The isotropy group at $x \in Q$ is $\Gamma_{V(x)}$.*

Theorem 1.10 shows that for Q and M as above, if $f: Q \to M$ is the inclusion map, then the Γ-map $f^*: U(\Gamma, Q) \to M$ defined by 1.7, is a homeomorphism.

Let C^n be the standard simplicial cone in $\mathbf{R}^n$ defined by the inequalities $x_i \geq 0$, $1 \leq i \leq n$. For any $x = (x_1, \ldots, x_n) \in C^n$ its codimension $C(x)$ is the number of x_i that are equal to zero. For each i, set $C_i = \{x \in C^n | x_i = 0\}$.

DEFINITION 1.11. A V-panelled n-manifold is a Hausdorff paracompact panelled space $\{Q, (Q_v)_{v \in V}\}$ in which each point has a neighborhood N and a homeomorphism of panelled spaces from $\{N(N_v)_{v \in V})$ to an open set of $\{C^n, (C_i)_{i=1,\ldots,n}\}$.

Note that Q is a manifold with boundary and $\partial Q = \bigcup\{Q_v, v \in V\}$. Also each Q_v is a manifold with boundary and Q_v is a $V - \{v\}$-panelled $n - 1$ manifold, i.e., Q_v has codimension one in Q. Inductively, then if S is a subset of V, Q_S is empty or is an $n - |S|$-dimensional $V - S$-panelled manifold; its boundary is $\partial Q_S = \bigcup\{Q_T: T \supset S, |T| = 1 + |S|\}$.

THEOREM 1.12 [**9**, p. 308]. *Let (Γ, V) be a Coxeter group and let Q be a connected V-panelled manifold with Γ-finite panel structure. Let $M = U(\Gamma, V)$. Then M is a manifold and (Γ, V) is a reflection group on M with fundamental chamber Q.*

THEOREM 1.13 [**17**, p. 1092]. *Let Γ be a discrete linear group generated by a set V of reflections in the faces of a "convex polyhedral cone" K (see [**17**]). Let $K^f = \{x \in K: \Gamma_{V(x)}$ is finite$\}$. Then the following statements hold.*

1. *$\bigcup_{g \in \Gamma} gK$ is a convex cone.*
2. *Γ acts properly on the interior Ω of this cone.*
3. *$\Omega \cap K = K^f$.*
4. *The canonical map from K^f to Ω/Γ is a homeomorphism.*
5. *Γ is a reflection group on Ω, and K^f is a closed chamber for Γ on Ω.*

Notice that the above theorem together with Theorem 1.10 implies that each linear Coxeter group is a Coxeter group.

THEOREM 1.14 [**17**, p. 1105]. *Every Coxeter group is isomoprhic to some linear Coxeter group.*

THEOREM 1.15 [**9**, p. 317]. *The virtual cohomological dimension of* Γ *is* $\leq$ *the dimension of* $K(\Gamma, V)$.

Serre in [**15**] proved that if the $\operatorname{Vcd}\Gamma < \infty$, then there is a finite dimensional Γ-complex $\xi\Gamma$ such that:

1. $\xi\Gamma$ is a proper Γ-complex, i.e., Γ acts cellularly, the stabilizer of each cell fixes that cell pointwise, and each point has finite isotropy group.

2. For each finite subgroup H of Γ, the fixed set $(\xi\Gamma)^H$ is contractible.

Properties 1 and 2 specify the Γ-homotopy type of the space $\xi\Gamma$, and we will write $\xi\Gamma$ for any finite dimensional Γ-complex satisfying properties 1 and 2.

An immediate consequence of Brown's collaring theorem (see [**6**]) is the following:

THEOREM 1.16. *Let Q be an n-dimensional compact topological manifold with boundary ∂Q. Then $(Q, \partial Q)$ is an NDR-pair (see* [**19**], *for the definition of NDR-pair).*

CHAPTER II. PROPERTIES OF ADMISSIBLE MANIFOLDS

Let (Γ, V) be a Coxeter group and $\{Q, (Q_v)_{v\in V}\}$ be an n-dimensional panelled manifold. We are going to study the Γ-manifolds $U(\Gamma, Q)$, when $\{Q, (Q_v)_{v\in V}\}$ has the property that Q_S is contractible if Γ_S is finite or Q_S is empty if Γ_S is infinite.

LEMMA 2.1. *Let (Γ, V) be a Coxeter group, and let H be a finite subgroup of Γ. Then there is a subset J of V such that J generates a finite subgroup Γ_J of Γ, and some conjugate of Γ_J contains H.*

PROOF. By [**17**], (Γ, V) is isomorphic to some linear Coxeter group, and also is a reflection group on Ω. By [**9**], Ω is Γ-homeomorphic to $U(\Gamma, C^f)$. Let p be any element in Ω. Since Ω is convex $1/|H|\sum_{h\in H} hp$ is an element of Ω. Let us call it ty, where $y \in C^f$ and $t\in\Gamma$. It follows from the definition of $U(\Gamma, C^f)$ that the isotropy group of ty is $t\Gamma_{V(y)}t^{-1}$. Since H fixes ty, we get $t\Gamma_{V(y)}t^{-1} \supset H$ as required.

Let (Γ, V) be a Coxeter group and Q be an n-dimensional connected V-panelled manifold with Γ-finite panel structure. By Theorem 1.12, $U(\Gamma, Q)$ is a manifold and (Γ, V) is a reflection group on $U(\Gamma, Q)$ with fundamental chamber Q. Let $V \supset J$ be such that $|\Gamma_J| < \infty$, and write $G = \Gamma_J$. We want to describe, as efficiently as possible, the fixed set $U(\Gamma, Q)^G$.

DEFINITION 2.2. A face gQ_S of $U(\Gamma, Q)^G$ is called a domain of $U(\Gamma, Q)^G$ if ${}^g\Gamma_S = G$.

Note that if Q_S is nonempty, then it is a domain of $U(\Gamma, Q)^G$.

LEMMA 2.3. *Assume $U(\Gamma, Q)^G$ is connected. Then $U(\Gamma, Q)^G = \bigcup\{wQ_T\colon wQ_T$ is a domain of $U(\Gamma, Q)^G\}$.*

PROOF. Each point of $U(\Gamma, Q)^G$ lies in some face wQ_T such that $w(\operatorname{Int} Q_T)$ is open in $U(\Gamma, Q)^G$, where $\operatorname{Int} Q_T$ denotes the interior of Q_T. By the theorem of invariance of domains all these open sets have the same dimension. The points of

$U(\Gamma, Q)^G$ having isotropy group $G = \Gamma_S$ form an open dense set in $U(\Gamma, Q)^G$, so at least one of these points is in each of the $w(\operatorname{Int} Q_T)$ above. And the result follows.

DEFINITION 2.4. Let (Γ, V) be a Coxeter group and Q be a compact n-dimensional V-panelled manifold. We say that Q is an admissible manifold if the following conditions are satisfied:

1. Q_S is contractible if Γ_S is finite.
2. Q_S is empty if Γ_S is infinite.

PROPOSITION 2.5. *Let (Γ, V) be a Coxeter group. Assume Q is a V-panelled admissible manifold. Assume that Q' is a Γ-finite V-panelled space and that for each S in V, $(Q'_S, \partial Q'_S)$ is an NDR-pair. Then there exists a V-panelled map $f\colon Q' \to Q$.*

PROOF. The map f: $Q' \to Q$ is constructed inductively on all the faces Q'_J, $V \supset J$ such that Γ_J is a finite group.

Use the fact that $(Q'_T, \partial Q'_T)$ is an NDR pair and the homotopy extension property in order to conclude that $f = f_\varnothing$: $Q' \to Q$ is the required map.

Let (Γ, V) be a Coxeter group. Let $\{Q, (Q_v)_{v\in V}\}$ and $\{Q', (Q'_v)_{v\in V}\}$ be V-panelled spaces.

DEFINITION 2.6. A map of V-panelled spaces F: $\{Q \times I, (Q_v \times I)_{v\in V}\} \to \{Q', (Q'_v)_{v\in V}\}$ is called a homotopy from $F|_{Q\times 0}$ to $F|_{Q\times 1}$.

Obviously if F is a homotopy between $f_0, f_1\colon \{Q, (Q_v)_{v\in V}\} \to \{Q', (Q'_v)_{v\in V}\}$, then $U(\Gamma, F)$ provides a Γ-equivariant homotopy from $U(\Gamma, f_0)$ to $U(\Gamma, f_1)$.

DEFINITION 2.7. Let (Γ, V) be a Coxeter group. Let $\{Q, (Q_v)_{v\in V}\}$ be a V-panelled manifold. Two faces gQ_S and wQ_T of $U(\Gamma, Q)$ are called adjacent if they have the same dimension and their intersection is a nonempty face of codimension one less than each.

The following is a key result.

PROPOSITION 2.8. *Let (Γ, V) be a right angled Coxeter group. Let $\{Q(Q_v)_{v\in V}\}$ be a Γ-finite V-panelled manifold. Let $G = \Gamma_J$ be a finite subgroup of Γ, where $V \supset J$. Let gQ_K be a domain of $U(\Gamma, G)^G$ that is adjacent to Q_J. Let T be the subset of V defined by $Q_T = Q_J \cap gQ_K$. Then*

a. *$T = J \cup V_g$.*

b. *$|T| = |J| + 1$, and Γ_T is a finite group and $K = J$.*

c. *$gQ_J = vQ_J$, where v is an element of V that commutes with all elements of T.*

PROOF. a. Follows from 1.6.

b. Follows from the definition of adjacent faces.

c. Follows from the definition of right angled Coxeter group and the fact that Γ_T and Γ_J are elementary abelian 2 groups, Γ_T/G has order 2 and is generated by $g\Gamma_J = v\Gamma_J$ where $T = J \cup \{v\}$. This completes the proof.

PROPOSITION 2.9. *Let S, T be subsets of V. Assume that $Q_S \cap gQ_T \neq \varnothing$ for some $g \in \Gamma$. Let $x \in Q_S \cap gQ_T$. Then $g \in \Gamma_{V(x)}, V(x) \supset S \cup T$ and $Q_S \cap Q_T \supset Q_s \cap gQ_T$.*

PROOF. Follows from the definition of $U(\Gamma, Q)$ and 1.6.

PROPOSITION 2.10. *Let (Γ, V) be a right angled Coxeter group and let $\{Q, (Q_v)_{v\in V}\}$ be an admissible V-panelled manifold. Let $G = \Gamma_J$ be a finite group, where J is some subset of V. Assume that $U(\Gamma, G)^G$ is connected. Let gQ_J be any domain of $U(\Gamma, Q)^G$. Then $gQ_J = v_1 \cdot v_2 \cdot v_3 \cdots v_m Q_J$ for some elements $v_i \in V$ that commute with all elements of J.*

PROOF. Since $U(\Gamma, Q)^G$ is a connected manifold, there is a finite sequence of domains of $U(\Gamma, Q)^G$ of the form $Q_J, g_1Q_J, g_2Q_J, \ldots, g_mQ_J = gQ_J$, each one adjacent to the next. Apply 2.8 and by induction if follows that $gQ_J = v_1 \cdot v_2 \cdots v_m Q_J$ and each v_i commutes with J.

Note that from the above proposition $g = v_1 \cdot v_2 \cdots v_m \cdot x$ where $v_i \in V - J$, $x \in \Gamma_J$, and each v_i commutes with x.

LEMMA 2.11. *Let (Γ, V) be a Coxeter group. Let $\{Q, (Q_v)_{v\in V}\}$ be an admissible V-panelled manifold. Let H be a finite subgroup of Γ such that $U(\Gamma, Q)^H$ is connected. Then there is an isotropy group K such that $U(\Gamma, Q)^H = U(\Gamma, Q)^K$.*

PROOF. Each isotropy group K has the form $g\Gamma_S g^{-1}$, where $V \supset S$ and Γ_S is a finite group; hence there are only a countable number of isotropy subgroups.

But $U(\Gamma, Q)^H = \bigcup\{U(\Gamma, Q)^K : K \supset H,\ K \text{ an isotropy group}\}$.

If $K \supset H$, $U(\Gamma, Q)^K$ is closed in $U(\Gamma, Q)^H$, and is a submanifold of $U(\Gamma, Q)^H$. But $U(\Gamma, Q)^H$ is connected. Now use the Baire category theorem in order to conclude that for some isotropy group $K \supset H$, $U(\Gamma, Q)^K = U(\Gamma, Q)^H$.

Let (Γ, V) be a Coxeter group. Let J be a subset of V.

Let $J^* = \{v \in V - J \text{ such that } v \text{ commutes with all elements of } J\}$.

Let $\{Q, (Q_v)_{v\in V}\}$ be a V-panelled space. Each Q_J has the structure of J^*-panelled space. This panel structure is defined by the rule $(Q_J)_v = Q_{J\cup\{v\}}$.

PROPOSITION 2.12. *Let (Γ, V) be a right angled Coxeter group and $\{Q, (Q_v)_{v\in V}\}$ be an admissible manifold. Then*

(a) *$U(\Gamma, Q)$ is a Γ-space of type $\xi\Gamma$.*

(b) *If J is a subset of V such that the group $G = \Gamma_J$ is finite, then the natural map $K: \Gamma_{J^*} \to N_\Gamma(G)/G$, sending g to gG, is an isomorphism.*

(c) *The natural map $F: U(\Gamma_{J^*}, Q_J) \to U(\Gamma, Q)^G$ sending $[g, x]$ to $[g, x]$, if $g \in \Gamma_{J^*}$ and $x \in Q_J$, is a K-equivariant homeomorphism.*

PROOF. (a) Is a consequence of (c) as follows: by [**9**, Theorem 10.1, part 4], $U(\Gamma_{J^*}, Q_J)$ is contractible. Now use 2.11, in order to conclude that $U(\Gamma, Q)$ has type $\xi\Gamma$.

So we only have to prove (b) and (c).

Now $G = \Gamma_J$ is a finite abelian group of order 2^q for some q.

$U(\Gamma, Q)$ is contractible by [**9**, Theorem 10.1], so $U(\Gamma, Q)^G$ is $Z/2$-acyclic by Smith theory (see [**3**, III Theorem 5.2]). In particular $U(\Gamma, Q)^G$ is connected. By 2.10, each domain of $U(\Gamma, Q)^G$ has the form gQ_J where $g \in \Gamma_{J^*}$. But if $g_1 \in N_\Gamma(G)$, g_1Q_J is also a domain in $U(\Gamma, Q)^G$. So for some $g \in \Gamma_{J^*}$, $g_1 = g \cdot x$ where $x \in \Gamma_J$. It follows that K is an epimorphism. If $g \in \ker(K)$, then $g \in \Gamma_{J^*} \cap \Gamma_J$. So by 1.2, $J^* \cap J \supset V_g$. Since $J^* \cap J = \varnothing$, $g = 1$. Therefore K is an isomorphism.

It is clear that F: $U(\Gamma_{J^*}, Q_J) \to U(\Gamma, Q)^G$ maps gQ_J homeomorphically onto ${}_gQ_J$ for all $g \in \Gamma_{J^*}$ and so yields a homeomorphism from $U(\Gamma_{J^*}, Q_J)$ to $U(\Gamma, Q)^G$. This completes the proof of 2.12.

PROPOSITION 2.13. *Let (Γ, V) be a right angled Coxeter group. Let Q be an admissible manifold. Let Q' be a V-panelled manifold with Γ-finite panel structure. Then any Γ-map $g: U(\Gamma, Q') \to U(\Gamma, Q)$ is Γ-homotopic to a Γ-map $U(\Gamma, f)$: $U(\Gamma, Q') \to U(\Gamma, Q)$, where f is a V-panelled map from Q' to Q.*

PROOF. By [**15**] and 1.15, $\operatorname{Vcd}\Gamma < \dim K(\Gamma, V)$. Since $\dim K(\Gamma, V)$ is finite, $\operatorname{Vcd}\Gamma < \infty$.

By 2.12, $U(\Gamma, Q)$ is of type $\xi\Gamma$. Since for each subset S of V such that Γ_S is finite, Q'_S is a finite dimensional manifold with boundary $\partial Q'_S$. So by Theorem 1.16, $(Q'_S, \partial Q'_S)$ is an NDR-pair for each subset S of V such that Γ_S is finite. Use 2.5, in order to get a V-panelled map f: $Q' \to Q$, and extend f to a Γ-map $U(\Gamma, f): U(\Gamma, Q') \to U(\Gamma, Q)$. By [**8**], g and $U(\Gamma, f)$ are in the same Γ-homotopy class.

PROPOSITION 2.14. *Suppose that Γ is a group generated by a set V of reflections on some manifold X, such that X/Γ is compact and X is of type $\xi\Gamma$. Let M be any Γ-manifold of type $\xi\Gamma$ such that M/Γ is compact. Then M^v has codimension one in M for each $v \in V$.*

PROOF. By 1.15, $\operatorname{Vcd}\Gamma \leq$ the dimension of $K(\Gamma, V)$. So there is a torsion free normal subgroup G of Γ of finite index.

Since X and M are of type $\xi\Gamma$, each G-isotropy group is trivial. So the projections $P: X \to X/G$ and $P': M \to M/G$ are regular covering maps and therefore X/G and M/G are of type $K(G, 1)$, and $H_*(G) \approx H_*(X/G) \approx H_*(M/G)$.

We conclude that $X/G = M/G$, and dimension of X = dimension of M. Since $M_\Gamma \approx X$, we have $X_H^v \approx M^v$, where $H = Z_G(v)$.

By [**8**], $X^v/Z_{G(v)} = (X/G)^{\langle v \rangle}$.

Since $[\Gamma: G] < \infty$, X/G is closed. But X/Γ is compact, so X/G is compact. So $X^v/Z_{G(v)}$ is compact. The same holds for M.

Use Proposition 7.5 and Corollary 7.6 in [**4**, p. 209], in order to conclude that $H_c^*(X^v: Z) \approx H_c^*(Z_G(v): ZZ_G(v)) \approx H_c^*(M^v: Z)$. Therefore, M^v has the same codimension of X^v for each $v \in V$.

CHAPTER III. RIGIDITY THEOREMS

Let (Γ, V) be a Coxeter group. Let M and M' be admissible manifolds. We are going to prove under some conditions on the group Γ and on M and M', that M and M' are equivariantly homeomorphic.

In this chapter we will assume that the reader is familiar with the concepts in surgery theory as exposed by C. T. C. Wall [**18**], and W. Browder [**3**].

LEMMA 3.1. *Let $(M^n, \partial M)$ be a compact contractible manifold. Then the structure set $\mathcal{S}(M^n, \partial M) = *$ if $n \neq 3$.*

PROOF. Since M is compact contractible, ∂M has the homology of the $n-1$ sphere.

Use the surgery exact sequence, the result $NM(M, \partial) = [M/\partial M, F/\mathrm{Top}] = [S^n, F/\mathrm{Top}]$ (see [**12**]), the fact that $\mathcal{S}(S^n) = *$, by the genralized Poincaré conjecture for $n \geq 5$, and Freedman for $n = 4$ (see [**11**]) in order to conclude that $\mathcal{S}(M^n, \partial M) = *$ if $n \neq 3$, $n \geq 5$.

The case when $n = 4$, follows from the 5-dimensional h-cobordism theorem [**13**, and **12**].

DEFINITION 3.2. Let Q be an admissible manifold. We say that Q is a strongly admissible manifold if it satisfies the following condition:

$**$ {If Q_S is a 3-dimensional panel of Q, then Q_S is homeomoprhic to I^3}.

PROPOSITION 3.3. *Let Q and Q' be strongly admissible manifolds. If $f: Q \to Q'$ is a map of n-dimensional compact V-paralled manifolds, then there is a homotopy F_t of V-panelled maps from $F = f$ to a homeomorphism g.*

PROOF. Let $\Psi = \{J, J$ subset of V such that $f_J: Q_J \to Q'_J$ is a homeomorphism, where $f_J = g|Q_j\}$.

Clearly Ψ is a nonempty set.

Let J be any subset of V such that J satisfies the following conditions:

1. If $K \supset J$ and $K \neq J$, then $K \in \Psi$.
2. $J \notin \Psi$.

It is clear that for any such J, $f|\partial Q_J: \partial Q_J \to \partial Q'_J$ is already a homeomorphism.

It is enough to show that there is a homotopy F_t of V-panelled mapps relative to $\bigcup_{K \in \Psi} Q_K$ from $F_0 = f$ to a map g such that $g|Q_J: Q_J \to Q'_J$ is a homeomorphism.

By repeated use of the homotopy extension property it is enough to show that $f|Q_J: Q_J \to Q'_J$ is homotopic relative to ∂Q_J, to a homeomorphism.

If the dimension of Q_J is not equal to 3, the result follows from Lemma 3.1, since $f|Q_J: (Q_J, \partial Q_J) \to (Q'_J, \partial Q'_J)$ is a homotopy equivalence of pairs.

Suppose that the dimension of Q_J is equal to 3. Since Q_J, and Q'_J are homeomorphic to I^3, we conclude that $f|Q_J: (I^3, \partial) \to (I^3, \partial)$ is homotopic relative to ∂I^3, to a homeomorphism h, where h is defined as follows ($f|_{\partial I^3}: S^2 \to S^2$ is an isomorphism, extend to $h = c(f|_{\partial I^3}): I^3 \to I^3$, and define $c(f|_{\partial I^3})(\alpha v) = \alpha((f|_{\partial I^3})(v))$, $0 \leq \alpha \leq 1$).

REMARK. If the Poincaré conjecture is true for $n = 3$, then admissible manifolds are strongly admissible, so Proposition 3.3, is valid for admissible manifolds.

PROPOSITION 3.4. *Let (Γ, V) be a right angled Coxeter group. Let Q and Q' be strongly admissible manifolds. Any Γ-map $f_0: U(\Gamma, Q) \to U(\Gamma, Q')$ is Γ-homotopic to a Γ-homeomorphism.*

PROOF. By 2.13, we may as well assume f_0 has the form $U(\Gamma, f)$ where $f: Q \to Q'$ is a map of V-panelled spaces. By Proposition 3.3, f is homotopic to a homeomorphism $g: Q \to Q'$ of V-panelled manifolds. So f_0 is Γ-homotopic to $U(\Gamma, g)$, which is obviously a Γ-homeomorphism.

DEFINITION 3.5 Let Γ be a Coxeter group acting locally smoothly and effectively on a manifold M. We say that M is a Coxeter manifold for Γ, if it satisfies the following conditions:

1. M/Γ is compact.
2. Γ acts properly discontinuously on M.
3. M^H is contractible for any finite subgroup H of Γ.
4. If M^H has dimension 3, then $M^H \approx R^3$.

REMARK 3.6. In [**9**], Michael Davis showed that certain Coxeter groups (Γ, V) act on contractible manifolds X^n ($\neq R^n$, $n \geq 4$) with compact quotients. These Davis groups have the following general properties:

1. The generators $v \in V$ act as reflections on X.
2. The action is properly discontinuously, locally smooth and effective.
3. The only entries of the Coxeter matrix are 1, 2, and ∞.
4. If S is a subset of V generating a finite subgroup G of Γ, then X^G is contractible.

Note that all Davis' examples are Coxeter manifolds.

DEFINITION 3.7. A Coxeter group that admits an action with the above properties 1–4 of 3.6 is called a right angled reflection group.

PROPOSITION 3.8. *Let (Γ, V) be a right angled reflection group. Let M be a Coxeter manifold for Γ. Then M is Γ-homeomoprhic to $U(\Gamma, Q)$, where Q is a fundamental domain.*

PROOF. By Remark 3.6, X is a Coxeter manifold for Γ of type $\xi\Gamma$. By 2.14, M^v has codimension one in M for each $v \in V$. We need only show that $M - M^v$ has exactly two components.

Assume that $n = \dim(M)$. Taking cohomology with compact support with coefficient in Z, the long exact sequence in cohomology is reduced to $0 \to Z \to H_c^n(M, M^v) \to Z \to 0$.

So $H_c^n(M, M^v) \approx Z \oplus Z$. But $H_c^n(M, M^v) \approx H_0(M - M^v)$. Therefore $M - M^v$ has exactly two components.

Let $M_0 = M - (\bigcup_{v \in V} M^v)$. Let C be a connected component of M_0. Define Q as the closure of C in M. Q is a fundamental domain, by 1.10, and $\Pi|_Q$ is a homeomorphism, where $\Pi: M \to M/\Gamma$. By hypothesis, M/Γ is compact, so Q is compact.

The natural panel structure $(Q_v)_{v \in V}$ of Q is defined as follows:

$Q_v = Q \cap M^v$.

Let $i: Q \to M$ be the inclusion of its fundamental domain. Then the induced map $i^*: U(\Gamma, Q) \to M$ is a Γ-equivariant homeomorphism (see [**9**]). So we conclude that $M \approx U(\Gamma, Q)$.

COROLLARY 3.9. *Let (Γ, V) be a right angled relfection group. Let M be a Coxeter manifold for Γ. Then M is Γ-homeömorphic to $U(\Gamma, Q)$, where Q is an admissible manifold.*

PROOF. By 3.8, M is Γ-homeomorphic to $U(\Gamma, Q)$, where Q is a compact fundamental domain. Let S be a subset of V. Write $G = \Gamma_s$ a finite subgroup of Γ. By 2.11, $U(\Gamma, Q)^G = U(\Gamma_{S^*}, Q_S)$. $U(\Gamma, Q)^H$ is contractible by [**9**, Theorem 10.1] Q_S is contractible. Therefore, we conclude that Q is an admissible manifold.

At this point, we have the following conjecture about Coxeter manifolds.

CONJECTURE. If M and M' are Coxeter manifolds, then M and M' are equivariantly homeomorphic.

We are going to prove this conjecture only in the case when the Coxeter group Γ is a right angled reflection group.

The following theorem is a key fact for our purpose.

THEOREM 3.10. *Let (Γ, V) be a right angled reflection group. Let M be a Coxeter manifold. Then M is Γ-homeomorphic to $U(\Gamma, Q)$, where Q is a compact strongly admissible manifold.*

PROOF. Since $\operatorname{Vcd}\Gamma < \infty$ and by hypothesis M^H is contractible for each finite subgroup H of Γ, so by [**8**], M is of type $\xi\Gamma$. By Corollary 3.9, M is Γ-homeomorphic to $U(\Gamma, Q)$, where Q is a compact admissible manifold.

Let H be any finite subgroup of Γ such that the dimension of M^H is equal to 3. By hypothesis $M^H = R^3$, so $U(\Gamma, Q)^H \approx R^3$. Using 2.11, $U(\Gamma, Q)^H = U(\Gamma, Q)^K$, where K is an isotropy group, $K \supset H$. Since K has the form gGg^{-1}, where $G = \Gamma_S$, $V \supset S$, $g \in \Gamma$, and Γ_S is a finite subgroup of Γ; and $U(\Gamma, Q)^K = gU(\Gamma, Q)^G$, we get that $U(\Gamma, Q)^K = gU(\Gamma_{S^*}, Q_S)$.

Claim. Q_S is homeomorphic to I^3.

Proof of claim. Since Q_S is a compact contractible 3-dimensional manifold with boundary in R^3, ∂Q_S is a 2-dimensional closed manifold of genus zero. So $\partial Q_S = S^2$, using the Schonflies theorem [**5**]. The closures of the components of $S^3 - S^2$ are 3-dimensional balls, so we conclude that Q_S is a 3-dimensional ball. Consequently Q is a strongly admissible manifold.

The following is our principal result:

THEOREM 3.11 (RIGIDITY THEOREM FOR COXETER MANIFOLDS). *Let (Γ, V) be a right angled reflection group. Any two Coxeter manifolds M and N are equivariantly homeomorphic. In fact, any Γ-map between them is Γ-homotopic to a homeomorphism.*

PROOF. Let M and M' be Coxeter manifolds. By 3.11, M, (respectively M') is Γ-homeomorphic to $U(\Gamma, Q)$, (respectively $U(\Gamma, Q')$), where Q and Q' are compact strongly admissible manifolds.

For each subset S of V, such that Γ_S is finite, Q_S and ∂Q_S are compact manifolds. By Theorem 1.16, $(Q_S, \partial Q_S)$ is an NDR-pair. Therefore there is a homotopy equivalence $f: Q \to Q'$ of V-panelled spaces. By 2.13, any Γ-map $g: U(\Gamma, Q) \to U(\Gamma, Q')$ is Γ-homotopic to a Γ-map $U(\Gamma, f): U(\Gamma, Q) \to U(\Gamma, Q')$. By 3.4, $U(\Gamma, f)$ is Γ-homotopic to a homeomorphism.

BIBLIOGRAPHY

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres IV, V, Hermann, Paris, 1968.
2. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
3. W. Browder, *Surgery on simple connected manifolds*, Springer-Verlag, New York, 1972.
4. K. S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1982.
5. M. Brown, *A proof of the generalized Schonflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76.
6. ——, *Locally flat embeddings of topological manifolds*, Ann. of Math. (2) **75** (1962), 331–341.
7. R. W. Carter, *Finite groups of Lie type, conjugacy class and complex characters*, Pure and Appl. Math., Ser. of Texts, Monographs & Tracts, Wiley-Interscience.
8. F. Connolly and T. Kozniewski, *Finiteness properties of classifying spaces of proper Γ-actions*, J. Pure Appl. Algebra (to appear).
9. M. W. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. (2) **117** (1983), 293–324.
10. F. T. Farrell and W. C. Hsiang, *The topological Euclidean space form problem*, Invent. Math. **45** (1978), 181–192.
11. M. Freedman, *The topology of four dimensional manifolds*, J. Differential Geometry **17** (1982), 357–453.
12. R. Kirby and L. Siebenmann, *Foundational essays on topological manifolds, smoothing and triangulations*, Ann. of Math. Studies, no. 88, Princeton Univ. Press, Princeton, N.J., 1977.
13. F. Quinn, *Ends of maps*. III: *Dimension* 4 *and* 5, J. Differential Geometry **17** (1982), 503–521.

14. C. P. Rouke and B. J. Sanderson, *Introduction to piecewise linear topology*, Springer-Verlag, New York, 1982.
15. J. P. Serre, *Cohomologie des groupes discrets*, Prospects in Math., Ann. of Math. Studies, no. 70, Princeton Univ. Press, Princeton, N.J., 1971, pp. 77–169.
16. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
17. E. B. Vinberg, *Discrete linear groups generated by reflections*, Math. USSR Izv. **5** (1971), 1083–1119.
18. C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1970.
19. G. W. Whitehead, *Elements of homotopy theory*, Springer-Verlag, New York, 1978.

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